Granična vrednost niza

$$A = \lim_{n\to\infty} a_n$$
$$\Leftrightarrow$$
$$\forall \varepsilon > 0\,,\ \exists n_0 \in \mathbb{N}\,,\ \forall n \geq n_0\,,\ |A - a_n| < \varepsilon$$

Određeni oblici

$\infty \cdot \infty = \infty\,,\qquad \infty + \infty = \infty\,,\qquad \infty^{\infty} = \infty\,,$

$\frac{1}{\pm 0} = \pm\infty\,,\qquad \frac{1}{\pm\infty} = 0\,,\qquad \frac{0}{\pm\infty} = 0\,,$

$0^{\infty} = 0\,,\qquad \frac{\infty}{\pm 0} = \pm\infty\,.$

Neodređeni oblici

$\infty - \infty\,,\ 0 \cdot \infty\,,\ \frac{0}{0}\,,\ \frac{\infty}{\infty}\,,\ 1^{\infty}\,,\ 0^{0}\,,\ \infty^{0}\,.$

Važne granične vrednosti

1. $\lim\limits_{n\to\infty} \dfrac{1}{n^{\alpha}} = \begin{cases} 0, & \alpha > 0 \\ 1, & \alpha = 0 \\ +\infty, & \alpha < 0 \end{cases}$

2. $\lim\limits_{n\to\infty} q^{n} = \begin{cases} 0, & |q| < 1 \\ 1, & q = 1 \\ +\infty, & q > 1 \end{cases}$

3. $\lim\limits_{\substack{n\to\infty \\ f(n)\to\pm\infty}} \left(1 + \dfrac{1}{f(n)}\right)^{g(n)} = e^{\lim\limits_{n\to\infty} \frac{g(n)}{f(n)}}$

4. $\lim\limits_{n\to\infty} \dfrac{n^{\alpha}}{a^{n}} = 0$, za $a > 1$

5. $\lim\limits_{n\to\infty} (q^{n} - r^{n}) = +\infty$, za $q > r > 1$

6. $\lim\limits_{n\to\infty} (n^{\alpha} - n^{\beta}) = +\infty$, za $\alpha > \beta > 0$

Izračunati:

$\frac{1{,}1}{100}$

4. $\lim\limits_{n\to\infty} \dfrac{1.1^{n}}{100^{n}} = \lim\limits_{n\to\infty} (0{,}011)^{n} = 0$

$\infty - \infty \qquad \to 0$

5. $\lim\limits_{n\to\infty} \dfrac{\sqrt{n^2+n+1} - \sqrt{n^2+3n-1}}{n+3}$

∞

Granična vrednost funkcije

$$\lim_{x\to a} f(x) = l \Leftrightarrow$$
$$\forall \epsilon > 0 \; \exists \delta > 0 \; (0 < |x-a| < \delta \Rightarrow |f(x) - l| < \epsilon)$$

Ako granične vrednosti postoje, važi:

$$\lim_{x\to a}(\alpha f(x) + \beta g(x)) = \alpha \lim_{x\to a} f(x) + \beta \lim_{x\to a} g(x)$$

$$\lim_{x\to a}(f(x) \cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x),} \quad \lim_{x\to a} g(x) \neq 0$$

$$\lim_{x\to a} h(f(x)) = h\left(\lim_{x\to a} f(x)\right), \; h \text{ neprekidno.}$$

Važne granične vrednosti

1. $\lim_{x\to 0} \frac{\sin x}{x} = 1$
2. $\lim_{x\to \pm\infty} \left(1 + \frac{1}{x}\right)^x = e$
3. $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$
4. $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$
5. $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$
6. $\lim_{x\to \infty} \arctan x = \pi/2$

Izračunati:

1. $\lim_{x\to\infty} \frac{x^2+3x}{2x^3-4x}$,
2. $\lim_{x\to\infty} \frac{x^2+3x}{2x^2-4x}$,
3. $\lim_{x\to\infty} \frac{x^3+3x}{2x^2-4x}$,
4. $\lim_{x\to 0} \frac{x^2+3x}{2x^3-4x}$,
5. $\lim_{x\to 0} \frac{x^2+3x}{2x^2-4x}$,
6. $\lim_{x\to 0} \frac{x^3+3}{2x^2-4x}$.

Granična vrednost funkcije

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$$\forall\epsilon > 0\ \exists\delta > 0\ (0 < |x-a| < \delta \Rightarrow |f(x) - l| < \epsilon)$$

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$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x),} \quad \lim_{x\to a} g(x) \neq 0$$

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6. $\lim_{x\to\infty}\arctan x = \pi/2$

Izračunati:

1. $\lim_{x\to 2}\frac{x^2-4}{x^2-5x+6}$,
2. $\lim_{x\to 4^+}\frac{\sqrt{x}-2}{\sqrt{x-4}}$,
3. $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}$.
4. $\lim_{x\to 0}\frac{\tan x}{x}$.
5. $\lim_{x\to 0}\frac{1-\cos x}{x^2}$.

1. Koristeći kalkulator rešiti jednačinu $1.16\,x^2 - 0.32\,x - 1.361364 = 0$.

2. Koristeći kalkulator izračunati graničnu vrednost $\lim_{n\to\infty}\left(\sqrt{n^2+4n+1} - \sqrt{n^2+n}\right)$.

3. Koristeći kalkulator izračunati graničnu vrednost $\lim_{x\to 0}\dfrac{1-\cos x}{x^2}$.

Izvod funkcije

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Ako funkcija ima prvi izvod u tački x, kažemo da je **diferencijabilna** u toj tački.

Osobine izvoda

1. $(\alpha f(x) + \beta g(x))' = \alpha f'(x) + \beta g'(x)$
2. $(f(x) \cdot g(x))' = f'(x)g(x) + f(x)g'(x)$
3. $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$
4. $(f(g(x)))' = f'(g(x))g'(x)$

Tablica izvoda

i) $c' = 0$

ii) $(x^n)' = nx^{n-1}$, $n \neq 0$

iii) $(\log_a x)' = \frac{1}{x \ln a}$

iv) $(a^x)' = a^x \ln a$

v) $(\sin x)' = \cos x$

vi) $(\cos x)' = -\sin x$

vii) $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

viii) $(\operatorname{arctg} x)' = \frac{1}{1+x^2}$

$\underbrace{ix)\ (\operatorname{tg} x)' = \frac{1}{\cos^2 x}}$

$\cos x \cdot \cos x = \cos^2 x$

Pokazati po definiciji da je $(x^2 - x + 1)' = 2x - 1$.

$$\lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - (x+\Delta x) + 1 - (x^2 - x + 1)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - x - \Delta x + 1 - x^2 + x - 1}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \left(\frac{\Delta x(2x-1)}{\Delta x} + \Delta x\right) = 2x - 1$$

Naći po definiciji izvod funkcije $f(x) = \sqrt{x}$

$$(\operatorname{tg}(x))' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\overbrace{\cos^2 x + \sin^2 x}^{1}}{\cos^2 x} = \frac{1}{\cos^2 x}$$

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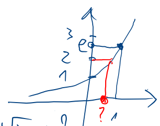
viii) $(\operatorname{arctg} x)' = \frac{1}{1+x^2}$

$$\left(x^{\frac{3}{2}}\right)' = \frac{3}{2} \cdot x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$$

1. $y = x\sqrt{x} - 3\sin x - \frac{1}{x} + \ln x^2, \quad y' =$

$\sqrt{x}\cdot\sqrt{x}$



$$(x\cdot\sqrt{x})' = x'\cdot\sqrt{x} + x\cdot(\sqrt{x})' = \sqrt{x} + x\cdot\frac{1}{2\sqrt{x}} = \sqrt{x} + \frac{1}{2}\sqrt{x} = \frac{3}{2}\sqrt{x}$$

$x > 0$

$$= \frac{3}{2\sqrt{x}}$$

g^x RPN

2 ENTER 1.5 CHS g^x

INFIX 0,3536

2 g^x 1.5 +/- =

2. $y = (x^2+x)^2 - e^{3+x} + \sqrt{2x}, \quad y' =$

$\ln 2 = \log_e 2 = ?$ 0,69

$e^{\boxed{?}} = 2$

$e = 2{,}72$

3. $y = \frac{x}{\sqrt{x}} + 2\cos x + \ln(2x), \quad y' =$

$x^{2}\cdot x^{\frac{1}{2}}$

$\sqrt{x} = x^{\frac{1}{2}}$

$$\frac{1}{x\sqrt{x}}\Big|_{x=2} = \frac{1}{2\sqrt{2}} \approx 0.35$$

$\sqrt{2} = 1.4$

4. $y = \frac{1}{x\sqrt{x}} - \frac{1}{e^{-x}} + \log_2 x, \quad y' = -\frac{3}{2}x^{-\frac{5}{2}} - e^x + \frac{1}{x\cdot\ln 2} = -\frac{3}{2x^2\sqrt{x}} - e^x + \frac{1}{x\ln 2}$

$= x^{-\frac{3}{2}} - e^x + \log_2 x$

$$x^{-\frac{3}{2}}\Big|_{x=2} = 2^{-\frac{3}{2}}$$

Naći prvi izvod funkcije $y = f(x)$ i uprostiti dobijeni izraz

a) $y = x^2 \cos x + \sin x$

b) $y = \dfrac{e^x}{x^2} + 2x \ln x$

c) $y = \operatorname{tg} x$

d) $y = \ln \operatorname{tg}\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right)$

e) $y = \dfrac{x^2+1}{x^2-x+1}$

f) $y = \dfrac{\ln(x-1)}{x^2-1}$

g) $y = \ln \dfrac{1+\operatorname{tg}\frac{x}{2}}{1-\operatorname{tg}\frac{x}{2}}$

$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ $\quad (fg)' = f'g + fg'$ $\quad (\ln x)' = \frac{1}{x}$

a) $y' = (x^2 \cos x + \sin x)' = (x^2 \cos x)' + (\sin x)' = (x^2)' \cos x + x^2 (\cos x)' + \cos x = 2x \cos x - x^2 \sin x + \cos x = (2x+1)\cos x - x^2 \sin x$

b) $y' = \left(\frac{e^x}{x^2}\right)' + (2x \ln x)' = \frac{e^x x^2 - e^x \cdot 2x}{x^{4\,3}} + 2\ln x + 2x\cdot\frac{1}{x} = e^x \frac{x-2}{x^3} + 2(\ln x + 1)$

$(\operatorname{tg} x)' = \frac{1}{\cos^2 x}$

e) $y' = \frac{2x(x^2-x+1) - (x^2+1)(2x-1)}{(x^2-x+1)^2} = \frac{2x^3 - 2x^2 + 2x - (2x^3 - x^2 + 2x - 1)}{(x^2-x+1)^2} = \frac{-x^2+1}{(x^2-x+1)^2}$

$\left(\frac{x}{2}\right)' = \left(\frac{1}{2}\cdot x\right)' = \frac{1}{2}$

$(f(g(x)))' = f'(g(x))\, g'(x)$

f) $y' = \frac{\frac{1}{x-1}(x-1)(x+1) - \ln(x-1)\cdot 2x}{(x^2-1)^2} = \frac{x+1-2x\ln(x-1)}{(x^2-1)^2}$

d) $y' = \frac{1}{\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)} \cdot \left(\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)\right)' = \frac{\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)}{\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)} \cdot \frac{1}{\cos^2\left(\frac{x}{2}+\frac{\pi}{4}\right)} \cdot \frac{1}{2} = \frac{1}{2\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)} = \frac{1}{\sin\left(x+\frac{\pi}{2}\right)} = \frac{1}{\sin x \cdot 0 + \cos x \cdot 1} = \frac{1}{\cos x}$

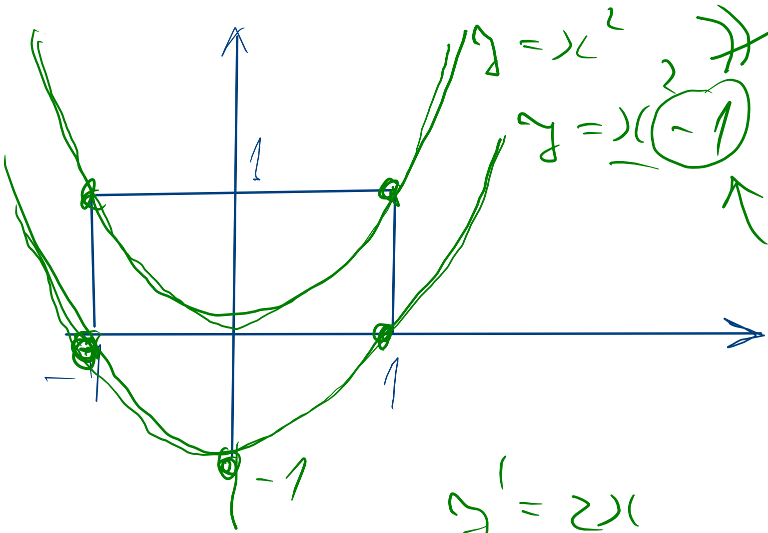
$\sin(2\alpha) = 2\sin\alpha\cos\alpha$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

g) $y' = \left(\ln\frac{1+\operatorname{tg}\frac{x}{2}}{1-\operatorname{tg}\frac{x}{2}}\right)' = \frac{1-\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}\frac{x}{2}} \cdot \frac{\frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2}\left(1-\operatorname{tg}\frac{x}{2}\right) + \left(1+\operatorname{tg}\frac{x}{2}\right)\frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2}}{\left(1-\operatorname{tg}\frac{x}{2}\right)^2} = \frac{2}{2\cos^2\frac{x}{2}\left(1-\operatorname{tg}^2\frac{x}{2}\right)} = \frac{1}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}} = \frac{1}{\cos x}$

$(a+b)(a-b)$

$\cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \cos(x)$

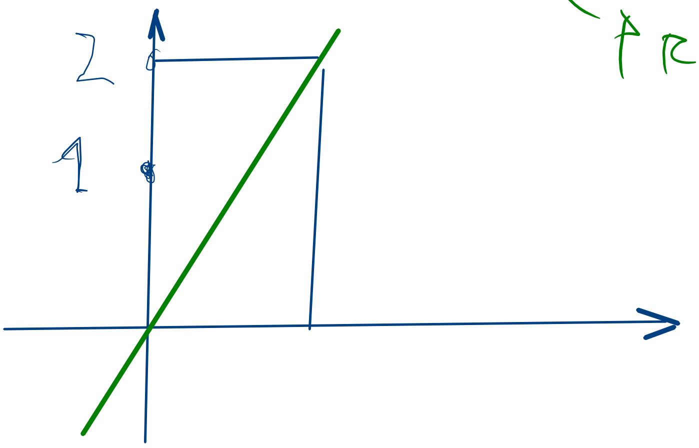


$y = x^2$ $\quad c = 0$

$y = x^2 - 1$ $\quad c = -1$

$y' = 2x$

PRVI IZVOD



$$\int 2x\,dx = x^2 + c$$

INTEGRACIONA KONSTANTA

PRIMITIVNA FUNKCIJA

NEODREĐENI INTEGRAL

(SKUP SVIH PRIMITIVNIH FUNKC.)

$$\int \cos x\,dx = \sin x + c$$

Naći prvi izvod funkcije $y = f(x)$ i uprostiti dobijeni izraz

a) $y = x^2\cos x + \sin x$

b) $y = \dfrac{e^x}{x^2} + 2x\ln x$

c) $y = \operatorname{tg} x$

d) $y = \ln\operatorname{tg}\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right)$

e) $y = \dfrac{x^2+1}{x^2-x+1}$

f) $y = \dfrac{\ln(x-1)}{x^2-1}$

g) $y = \ln\dfrac{1+\operatorname{tg}\frac{x}{2}}{1-\operatorname{tg}\frac{x}{2}}$

a)
$$\begin{aligned}
y' &= (x^2\cos x + \sin x)' \\
1.\Rightarrow &= (x^2\cos x)' + (\sin x)' \\
\text{v)},2.\Rightarrow &= (x^2)'\cos x + x^2(\cos x)' + \cos x \\
\text{ii), vi)}\Rightarrow &= 2x\cos x + x^2(-\sin x) + \cos x \\
&= (2x+1)\cos x - x^2\sin x
\end{aligned}$$

b)
$$\begin{aligned}
y' &= (x^{-2}e^x + 2x\ln x)' \\
1.\Rightarrow &= (x^{-2}e^x)' + (2x\ln x)' \\
2.\Rightarrow &= (x^{-2})'e^x + x^{-2}(e^x)' + (2x)'\ln x + 2x(\ln x)' \\
\text{ii), iii), iv)}\Rightarrow &= -2x^{-3}e^x + x^{-2}e^x + 2\ln x + 2x\cdot\frac{1}{x} \\
&= \frac{x-2}{x^3}e^x + 2\ln x + 2
\end{aligned}$$

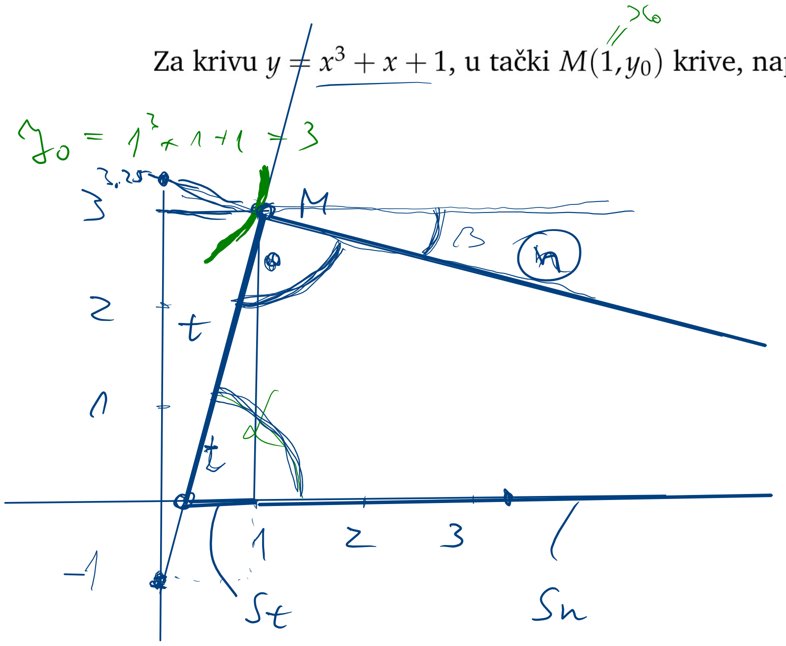
c)
$$\begin{aligned}
y' &= \left(\frac{\sin x}{\cos x}\right)' \\
3.\Rightarrow &= \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x} \\
\text{v), vi)}\Rightarrow &= \frac{\cos^2 x - (-\sin^2 x)}{\cos^2 x} = \frac{1}{\cos^2 x}
\end{aligned}$$

d)
$$\begin{aligned}
y' &= \left(\ln\operatorname{tg}\left(\tfrac{x}{2}+\tfrac{\pi}{4}\right)\right)' \\
\text{iii), 4.}\Rightarrow &= \frac{1}{\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)}\left(\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)\right)' \\
\text{c), 4.}\Rightarrow &= \frac{\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)}{\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)}\cdot\frac{1}{\cos^2\left(\frac{x}{2}+\frac{\pi}{4}\right)}\cdot\left(\frac{x}{2}+\frac{\pi}{4}\right)' \\
\text{i), ii), iii)}\Rightarrow &= \frac{1}{\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)}\cdot\frac{1}{2} \\
&= \frac{1}{\sin\left(x+\frac{\pi}{2}\right)} \\
&= \frac{1}{\cos x}
\end{aligned}$$

e) $-\dfrac{x^2-1}{(x^2-x+1)^2}$ f) $\dfrac{x+1-2x\ln(x-1)}{(x^2-1)^2}$ g) $\dfrac{1}{\cos x}$

Za krivu $y = x^3 + x + 1$, u tački $M(1, y_0)$ krive, napisati jednačinu tangente i normale.

$y_0 = 1^3 + 1 + 1 = 3$



$t: \ y = kx + n$

$M(1, 3)$

$k = tg\,\alpha = y'(1)$

$y' = 3x^2 + 1$ $\quad y'(1) = 3 \cdot 1^2 + 1 = 4 = k$

$y = 4x + n$ $\quad 3 = 4 \cdot 1 + n \rightarrow n = -1$

$t: \ y = 3x - 1$

$n: \ y = k_1 \cdot x + n_1$

$k_1 = -\frac{1}{k}$!!

$y = -\frac{1}{4} \cdot x + n_1$

$3 = -\frac{1}{4} \cdot 1 + n_1$ $\quad n_1 = 3 + \frac{1}{4} = \frac{13}{4}$

$= 3.25$

Za krivu $y = x^3 + x + 1$, u tački $M(1, y_0)$ krive, napisati jednačinu tangente i normale.
Nađimo prvi izvod posmatrane funkcije $y' = (x^3 + x + 1)' = 3x^2 + 1$. Za $x_0 = 1$ je $y_0 = y(x_0) = 1^3 + 1 + 1 = 3$ i $y'(x_0) = 3 \cdot 1^2 + 1 = 4$.
Jednačina tangente t glasi $y - y_0 = y'(x_0)(x - x_0)$, odnosno

$$t : y - 3 = 4(x - 1) \Leftrightarrow y = 4x - 4 + 3 = 4x - 1\,.$$

Jednačina normale n glasi $y - y_0 = -\frac{1}{y'(x_0)}(x - x_0)$, odnosno

$$n : y - 3 = -\frac{1}{4}(x - 1) \Leftrightarrow y = -\frac{1}{4}x + \frac{1}{4} + 3 = -\frac{1}{4}x + \frac{13}{4}\,.$$

NEODREĐENI INTEGRAL

$$\int \frac{dx}{\cos x} = \int \underbrace{\frac{1}{\cos x}}_{f(x)} dx$$

$+, -, *, \div, \sqrt{\ }, x^{x}, e^{x}, \ln x, \arcsin x, \operatorname{arctg} x, \dots$

$$\int \frac{\sin x}{x} dx = ?$$

POSTOJI $\mathrm{Si}(x)$

NEMA KONAČNE FORMULE PREKO ELEMENTARNIH FUNKCIJA

$$\int e^{x^2} dx$$

POSTOJI $\underbrace{\mathrm{erf}(x)}_{\text{PREKO}}$

$$(\ln x)' = \frac{1}{x}$$

$(0, +\infty)$ $\quad$ $\cancel{(-\infty, 0)} \cup (0, +\infty)$

$$(x^n)' = n \cdot x^{n-1}, \quad n \neq 0$$

$$\int n \cdot x^{n-1} dx = x^n + c, \quad n \neq 0$$

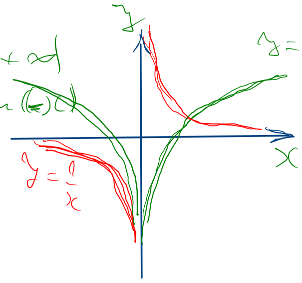
$$\int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$$

$$\int \frac{1}{x} dx = \ln|x| + c$$

$\mathcal{D} = \mathbb{R} \setminus \{0\}$

$\mathcal{D} = (0, +\infty)$

$y = \ln(-x)$



$$= \begin{cases} \ln x, & x > 0 \\ \ln(-x), & x < 0 \end{cases}$$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$