

Granična vrednost niza

$$A = \lim_{n \rightarrow \infty} a_n$$
$$\Leftrightarrow$$

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, |A - a_n| < \varepsilon$$

Određeni oblici

$$\infty \cdot \infty = \infty, \quad \infty + \infty = \infty, \quad \infty^\infty = \infty,$$

$$\frac{1}{\pm 0} = \pm \infty, \quad \frac{1}{\pm \infty} = 0, \quad \frac{0}{\pm \infty} = 0,$$

$$0^\infty = 0, \quad \frac{\infty}{\pm 0} = \pm \infty.$$

Neodređeni oblici

$$\infty - \infty, \quad 0 \cdot \infty, \quad \frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 1^\infty, \quad 0^0, \quad \infty^0.$$

Izračunati:

$$4. \lim_{n \rightarrow \infty} \frac{1.1^n}{100^n}$$

$$5. \lim_{n \rightarrow \infty} \frac{\sqrt{n^2 + n + 1} - \sqrt{n^2 + 3n - 1}}{n + 3}$$

Važne granične vrednosti

$$1. \lim_{n \rightarrow \infty} \frac{1}{n^\alpha} = \begin{cases} 0, & \alpha > 0 \\ 1, & \alpha = 0 \\ +\infty, & \alpha < 0 \end{cases}$$

$$2. \lim_{n \rightarrow \infty} q^n = \begin{cases} 0, & |q| < 1 \\ 1, & q = 1 \\ +\infty, & q > 1 \end{cases}$$

$$3. \lim_{\substack{n \rightarrow \infty \\ f(n) \rightarrow \pm \infty}} \left(1 + \frac{1}{f(n)}\right)^{g(n)} = e^{\lim_{n \rightarrow \infty} \frac{g(n)}{f(n)}}$$

$$4. \lim_{n \rightarrow \infty} \frac{n^\alpha}{a^n} = 0, \text{ za } a > 1$$

$$5. \lim_{n \rightarrow \infty} (q^n - r^n) = +\infty, \text{ za } q > r > 1$$

$$6. \lim_{n \rightarrow \infty} (n^\alpha - n^\beta) = +\infty, \text{ za } \alpha > \beta > 0$$

Granična vrednost funkcije

$$\lim_{x \rightarrow a} f(x) = l \Leftrightarrow$$

$$\forall \epsilon > 0 \exists \delta > 0 (0 < |x - a| < \delta \Rightarrow |f(x) - l| < \epsilon)$$

Ako granične vrednosti postoje, važi:

$$\lim_{x \rightarrow a} (\alpha f(x) + \beta g(x)) = \alpha \lim_{x \rightarrow a} f(x) + \beta \lim_{x \rightarrow a} g(x)$$

$$\lim_{x \rightarrow a} (f(x) \cdot g(x)) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$$

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}, \quad \lim_{x \rightarrow a} g(x) \neq 0$$

$$\lim_{x \rightarrow a} h(f(x)) = h\left(\lim_{x \rightarrow a} f(x)\right), \quad h \text{ neprekidno.}$$

Važne granične vrednosti

$$1. \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$2. \lim_{x \rightarrow \pm\infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$3. \lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}} = e$$

$$4. \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

$$5. \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$$

$$6. \lim_{x \rightarrow \infty} \arctan x = \pi/2$$

Izračunati:

$$1. \lim_{x \rightarrow \infty} \frac{x^2 + 3x}{2x^3 - 4x},$$

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$$3. \lim_{x \rightarrow \infty} \frac{x^3 + 3x}{2x^2 - 4x},$$

$$4. \lim_{x \rightarrow 0} \frac{x^2 + 3x}{2x^3 - 4x},$$

$$5. \lim_{x \rightarrow 0} \frac{x^2 + 3x}{2x^2 - 4x},$$

$$6. \lim_{x \rightarrow 0} \frac{x^3 + 3}{2x^2 - 4x}.$$

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Izračunati:

$$1. \lim_{x \rightarrow 2} \frac{x^2 - 4}{x^2 - 5x + 6},$$

$$2. \lim_{x \rightarrow 4^+} \frac{\sqrt{x} - 2}{\sqrt{x} - 4},$$

$$3. \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x}.$$

$$4. \lim_{x \rightarrow 0} \frac{\tan x}{x}.$$

$$5. \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}.$$

1. Koristeći kalkulator rešiti jednačinu $1.16x^2 - 0.32x - 1.361364 = 0$.

2. Koristeći kalkulator izračunati graničnu vrednost $\lim_{n \rightarrow \infty} (\sqrt{n^2 + 4n + 1} - \sqrt{n^2 + n})$.

3. Koristeći kalkulator izračunati graničnu vrednost $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$.

Izvod funkcije

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Ako funkcija ima prvi izvod u tački x , kažemo da je **diferencijabilna** u toj tački.

Osobine izvoda

$$1. (\alpha f(x) + \beta g(x))' = \alpha f'(x) + \beta g'(x)$$

$$2. (f(x) \cdot g(x))' = f'(x)g(x) + f(x)g'(x)$$

$$3. \left(\frac{f(x)}{g(x)} \right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

$$4. (f(g(x)))' = f'(g(x))g'(x)$$

Tablica izvoda

$$\text{i) } c' = 0$$

$$\text{ii) } (x^n)' = nx^{n-1}, n \neq 0$$

$$\text{iii) } (\log_a x)' = \frac{1}{x \ln a}$$

$$\text{iv) } (a^x)' = a^x \ln a$$

$$\text{v) } (\sin x)' = \cos x$$

$$\text{vi) } (\cos x)' = -\sin x$$

$$\text{vii) } (\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$\text{viii) } (\operatorname{arctg} x)' = \frac{1}{1+x^2}$$

Pokazati po definiciji da je $(x^2 - x + 1)' = 2x - 1$.

$$\begin{aligned} & \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - (x + \Delta x) + 1 - (x^2 - x + 1)}{\Delta x} = \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - x - \Delta x + 1 - x^2 + x - 1}{\Delta x} = \\ &= \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta x(2x - 1)}{\Delta x} + \Delta x \right) = 2x - 1 \end{aligned}$$

Naći po definiciji izvod funkcije $f(x) = \sqrt{x}$

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$$4. (f(g(x)))' = f'(g(x))g'(x)$$

$$1. y = x\sqrt{x} - 3\sin x - \frac{1}{x} + \ln x^2, \quad y' =$$

$$2. y = (x^2 + x)^2 - e^{3+x} + \sqrt{2x}, \quad y' =$$

$$3. y = \frac{x}{\sqrt{x}} + 2\cos x + \ln(2x), \quad y' =$$

$$4. y = \frac{1}{x\sqrt{x}} - \frac{1}{e^{-x}} + \log_2 x, \quad y' =$$

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Naći prvi izvod funkcije $y = f(x)$ i uprostiti dobijeni izraz

a) $y = x^2 \cos x + \sin x$

e) $y = \frac{x^2 + 1}{x^2 - x + 1}$

b) $y = \frac{e^x}{x^2} + 2x \ln x$

f) $y = \frac{\ln(x-1)}{x^2-1}$

c) $y = \operatorname{tg} x$

d) $y = \ln \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right)$

g) $y = \ln \frac{1 + \operatorname{tg} \frac{x}{2}}{1 - \operatorname{tg} \frac{x}{2}}$

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a) $y' = (x^2 \cos x + \sin x)'$

1. $\Rightarrow = (x^2 \cos x)' + (\sin x)'$

v), 2. $\Rightarrow = (x^2)' \cos x + x^2 (\cos x)' + \cos x$

ii), vi) $\Rightarrow = 2x \cos x + x^2 (-\sin x) + \cos x$
 $= (2x + 1) \cos x - x^2 \sin x$

b) $y' = (x^{-2} e^x + 2x \ln x)'$

1. $\Rightarrow = (x^{-2} e^x)' + (2x \ln x)'$

2. $\Rightarrow = (x^{-2})' e^x + x^{-2} (e^x)' + (2x)' \ln x + 2x (\ln x)'$

ii), iii), iv) $\Rightarrow = -2x^{-3} e^x + x^{-2} e^x + 2 \ln x + 2x \cdot \frac{1}{x}$
 $= \frac{x-2}{x^3} e^x + 2 \ln x + 2$

c) $y' = \left(\frac{\sin x}{\cos x} \right)'$

3. $\Rightarrow = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x}$

v), vi) $\Rightarrow = \frac{\cos^2 x - (-\sin^2 x)}{\cos^2 x} = \frac{1}{\cos^2 x}$

d) $y' = \left(\ln \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right)'$

iii), 4. $\Rightarrow = \frac{1}{\operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right)} \left(\operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right)'$

c), 4. $\Rightarrow = \frac{\cos \left(\frac{x}{2} + \frac{\pi}{4} \right)}{\sin \left(\frac{x}{2} + \frac{\pi}{4} \right)} \cdot \frac{1}{\cos^2 \left(\frac{x}{2} + \frac{\pi}{4} \right)} \cdot \left(\frac{x}{2} + \frac{\pi}{4} \right)'$

i), ii), iii) $\Rightarrow = \frac{1}{\sin \left(\frac{x}{2} + \frac{\pi}{4} \right) \cos \left(\frac{x}{2} + \frac{\pi}{4} \right)} \cdot \frac{1}{2}$
 $= \frac{1}{\sin \left(x + \frac{\pi}{2} \right)}$
 $= \frac{1}{\cos x}$

e) $-\frac{x^2 - 1}{(x^2 - x + 1)^2}$

f) $\frac{x+1-2x \ln(x-1)}{(x^2-1)^2}$

g) $\frac{1}{\cos x}$

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Nađimo prvi izvod posmatrane funkcije $y' = (x^3 + x + 1)' = 3x^2 + 1$. Za $x_0 = 1$ je $y_0 = y(x_0) = 1^3 + 1 + 1 = 3$ i $y'(x_0) = 3 \cdot 1^2 + 1 = 4$.

Jednačina tangente t glasi $y - y_0 = y'(x_0)(x - x_0)$, odnosno

$$t : y - 3 = 4(x - 1) \Leftrightarrow y = 4x - 4 + 3 = 4x - 1.$$

Jednačina normale n glasi $y - y_0 = -\frac{1}{y'(x_0)}(x - x_0)$, odnosno

$$n : y - 3 = -\frac{1}{4}(x - 1) \Leftrightarrow y = -\frac{1}{4}x + \frac{1}{4} + 3 = -\frac{1}{4}x + \frac{13}{4}.$$