Dual od duala

Za problem lineanog programiranja

$$
\begin{array}{rrcrcrcl}
\zeta = & 4x_1 & + & x_2 & + & 3x_3 & \to & \max \\
& x_1 & + & 4x_2 & & & \leq & 1 \\
& 3x_1 & - & x_2 & + & x_3 & \leq & 3 \\
& \multicolumn{7}{l}{x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0}
\end{array} \tag{9}
$$

dual je

$$
\begin{array}{rrcrcl}
\xi = & y_1 & + & 3y_2 & \to & \min \\
& y_1 & + & 3y_2 & \geq & 4 \\
& 4y_1 & - & y_2 & \geq & 1 \\
& & & y_2 & \geq & 3 \\
& \multicolumn{5}{l}{y_1 \geq 0, \quad y_2 \geq 0.}
\end{array} \tag{10}
$$

Dual u standardnom obliku je

$$
\begin{array}{rrcrcl}
-\xi = & -y_1 & - & 3y_2 & \to & \max \\
& -y_1 & - & 3y_2 & \leq & -4 \\
& -4y_1 & + & y_2 & \leq & -1 \\
& & - & y_2 & \leq & -3 \\
& \multicolumn{5}{l}{y_1 \geq 0, \quad y_2 \geq 0.}
\end{array} \tag{10a}
$$

Prelazak na dual ($x \leftrightarrow y$) daje

$$
\begin{array}{rrcrcrcl}
-\zeta = & -4x_1 & - & x_2 & - & 3x_3 & \to & \min \\
& -x_1 & - & 4x_2 & & & \geq & -1 \\
& -3x_1 & + & x_2 & - & x_3 & \geq & -3 \\
& \multicolumn{7}{l}{x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0,}
\end{array} \tag{9a}
$$

što je ekvivalentno sa (9).

Rečnik u matričnom obliku - izvođenje

(18) $\zeta = c^T x = [c_B^T \, c_N^T] \begin{bmatrix} x_B \\ x_N \end{bmatrix} = c_B^T x_B + c_N^T x_N$

(19) $b = Ax = [B \, N] \begin{bmatrix} x_B \\ x_N \end{bmatrix} = Bx_B + Nx_N \quad / \cdot B^{-1}$

Za matricu B postoji B^{-1}. Rečnik se dobija rešavanjem x_B i ζ preko x_N iz (19) i (18):

(19a) $B^{-1} b = x_B + B^{-1} N x_N \Rightarrow x_B = B^{-1} b - \left(B^{-1} N\right) x_N \Rightarrow$

(19b) $\zeta = c_B^T \left(B^{-1} b - \left(B^{-1} N\right) x_N\right) + c_N^T x_N = c_B^T B^{-1} b - \left(c_B^T \left(B^{-1} N\right) - c_N^T\right) x_N =$

$\left(\text{znajući} \quad (A \pm B)^T = A^T \pm B^T \quad \text{i} \quad (A\,B)^T = B^T A^T \quad \text{i} \quad A^{T^T} = A\right)$

(19c) $= c_B^T B^{-1} b - \left(\left(B^{-1} N\right)^T c_B - c_N\right)^T x_N$

Iz (19c) i (19a) dobijamo

(20)
$$\begin{aligned} \zeta &= c_B^T B^{-1} b - \left(\left(B^{-1} N\right)^T c_B - c_N\right)^T x_N \\ x_B &= B^{-1} b - B^{-1} N x_N \,. \end{aligned}$$

Kad uvedemo oznake

(21) $x_B^* = B^{-1} b, \; \zeta^* = c_B^T B^{-1} b,$

(23) $z_N^* = \left(B^{-1} N\right)^T c_B - c_N,$

dobijamo rečnike primara i duala u matričnom zapisu

	Primar	Dual
(24)	$\zeta = \zeta^* - (z_N^*)^T x_N$	$-\xi = -\zeta^* - (x_B^*)^T z_B$
	$x_B = x_B^* - \left(B^{-1} N\right) x_N$	$z_N = z_N^* + \left(B^{-1} N\right)^T z_B.$

Iz (24) se vidi osobina negativnog transponovanja i dokaz Teoreme jake dualnosti.

Permutovanje i transponovanje

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix} \cdot \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \\ b_{3,1} & b_{3,2} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{1,1}b_{1,1} + a_{1,2}b_{2,1} + a_{1,3}b_{3,1} & a_{1,1}b_{1,2} + a_{1,2}b_{2,2} + a_{1,3}b_{3,2} \\ a_{2,1}b_{1,1} + a_{2,2}b_{2,1} + a_{2,3}b_{3,1} & a_{2,1}b_{1,2} + a_{2,2}b_{2,2} + a_{2,3}b_{3,2} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{1,2}b_{2,1} + a_{1,1}b_{1,1} + a_{1,3}b_{3,1} & a_{1,2}b_{2,2} + a_{1,1}b_{1,2} + a_{1,3}b_{3,2} \\ a_{2,2}b_{2,1} + a_{2,1}b_{1,1} + a_{2,3}b_{3,1} & a_{2,2}b_{2,2} + a_{2,1}b_{1,2} + a_{2,3}b_{3,2} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{1,2} & a_{1,1} & a_{1,3} \\ a_{2,2} & a_{2,1} & a_{2,3} \end{bmatrix} \cdot \begin{bmatrix} b_{2,1} & b_{2,2} \\ b_{1,1} & b_{1,2} \\ b_{3,1} & b_{3,2} \end{bmatrix}$$

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}^{T^T} = \begin{bmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \\ a_{1,3} & a_{2,3} \end{bmatrix}^T = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}$$

$$\left(\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix} \cdot \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \\ b_{3,1} & b_{3,2} \end{bmatrix} \right)^T =$$

$$= \begin{bmatrix} a_{1,1}b_{1,1} + a_{1,2}b_{2,1} + a_{1,3}b_{3,1} & a_{1,1}b_{1,2} + a_{1,2}b_{2,2} + a_{1,3}b_{3,2} \\ a_{2,1}b_{1,1} + a_{2,2}b_{2,1} + a_{2,3}b_{3,1} & a_{2,1}b_{1,2} + a_{2,2}b_{2,2} + a_{2,3}b_{3,2} \end{bmatrix}^T =$$

$$= \begin{bmatrix} a_{1,1}b_{1,1} + a_{1,2}b_{2,1} + a_{1,3}b_{3,1} & a_{2,1}b_{1,1} + a_{2,2}b_{2,1} + a_{2,3}b_{3,1} \\ a_{1,1}b_{1,2} + a_{1,2}b_{2,2} + a_{1,3}b_{3,2} & a_{2,1}b_{1,2} + a_{2,2}b_{2,2} + a_{2,3}b_{3,2} \end{bmatrix} =$$

$$= \begin{bmatrix} b_{1,1}a_{1,1} + b_{2,1}a_{1,2} + b_{3,1}a_{1,3} & b_{1,1}a_{2,1} + b_{2,1}a_{2,2} + b_{3,1}a_{2,3} \\ b_{1,2}a_{1,1} + b_{2,2}a_{1,2} + b_{3,2}a_{1,3} & b_{1,2}a_{2,1} + b_{2,2}a_{2,2} + b_{3,2}a_{2,3} \end{bmatrix} =$$

$$= \begin{bmatrix} b_{1,1} & b_{2,1} & b_{3,1} \\ b_{1,2} & b_{2,2} & b_{3,2} \end{bmatrix} \cdot \begin{bmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \\ a_{1,3} & a_{2,3} \end{bmatrix} =$$

$$= \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \\ b_{3,1} & b_{3,2} \end{bmatrix}^T \cdot \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}^T$$

Množenje po blokovima

$$A=\begin{bmatrix}a&b\\c&d\end{bmatrix},\ B=\begin{bmatrix}e&f\\g&h\end{bmatrix},\ C=\begin{bmatrix}i&j\\k&l\end{bmatrix},\ D=\begin{bmatrix}m&n\\o&p\end{bmatrix},\ O=\begin{bmatrix}0&0\\0&0\end{bmatrix},\ I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$\begin{bmatrix}a&b&1&0\\c&d&0&1\\0&0&e&f\\0&0&g&h\end{bmatrix}\cdot\begin{bmatrix}i&j&1&0\\k&l&0&1\\0&0&m&n\\0&0&o&p\end{bmatrix}=\left[\begin{array}{cc|cc}a&b&1&0\\c&d&0&1\\\hline 0&0&e&f\\0&0&g&h\end{array}\right]\cdot\left[\begin{array}{cc|cc}i&j&1&0\\k&l&0&1\\\hline 0&0&m&n\\0&0&o&p\end{array}\right]=\left[\begin{array}{c|c}A&I\\\hline O&B\end{array}\right]\cdot\left[\begin{array}{c|c}C&I\\\hline O&D\end{array}\right]=$$

$$=\left[\begin{array}{c|c}AC+IO&AI+ID\\\hline OC+BO&OI+BD\end{array}\right]=\left[\begin{array}{c|c}AC&A+D\\\hline O&BD\end{array}\right]=\left[\begin{array}{cc|cc}ai+bk&aj+bl&a+m&b+n\\ci+dk&cj+dl&c+o&d+p\\\hline 0&0&em+fo&en+fp\\0&0&gm+ho&gn+hp\end{array}\right]$$

Invertovanje matrice

$$\begin{bmatrix}1&0\\1&3\end{bmatrix}^{-1}=?$$

$$\begin{bmatrix}1&0\\1&3\end{bmatrix}^{-1}=(1\cdot 3-1\cdot 0)^{-1}\begin{bmatrix}3&-1\\0&1\end{bmatrix}^{T}=\frac{1}{3}\begin{bmatrix}3&0\\-1&1\end{bmatrix}=\begin{bmatrix}1&0\\-1/3&1/3\end{bmatrix}$$

$$\begin{bmatrix}1&0&2\\2&1&2\\3&2&3\end{bmatrix}^{-1}=?$$

$$\left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\2&1&2&0&1&0\\3&2&3&0&0&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\0&1&-2&-2&1&0\\0&2&-3&-3&0&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\0&1&-2&-2&1&0\\0&0&1&1&-2&1\end{array}\right]\sim$$

$$\sim\left[\begin{array}{ccc|ccc}1&0&0&-1&4&-2\\0&1&0&0&-3&2\\0&0&1&1&-2&1\end{array}\right]\Rightarrow\begin{bmatrix}1&0&2\\2&1&2\\3&2&3\end{bmatrix}^{-1}=\begin{bmatrix}-1&4&-2\\0&-3&2\\1&-2&1\end{bmatrix}$$

$$\left[\begin{array}{cc|cc}1&0&1&0\\1&3&0&1\end{array}\right]\sim\left[\begin{array}{cc|cc}1&0&1&0\\0&3&-1&1\end{array}\right]\sim\left[\begin{array}{cc|cc}1&0&1&0\\0&1&-1/3&1/3\end{array}\right]$$

Data je mreža transporta sa čvorovima $\mathcal{N} = \{a,b,c,d,e\}$ i granama $\mathcal{A} = \{ab,ac,ad,bc,be,cd,ce,de\}$. Zalihe su redom $\{5,3,2,-4,-6\}$, a cene transporta redom $\{4,3,2,3,5,2,3,2\}$.

Problem linearnog programiranja koji odgovara nalaženju najjeftinijeg transporta je $c^T x \to \min$, $Ax = -b$, $x \geq 0$.

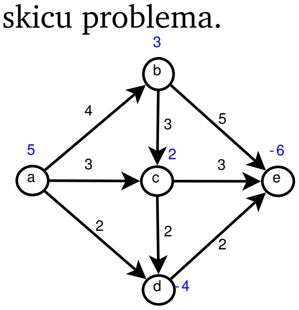
Odrediti matrice A, b i c.

$$A = \begin{bmatrix} -1 & -1 & -1 & & & & & \\ 1 & & & -1 & -1 & & & \\ & 1 & & 1 & & -1 & -1 & \\ & & 1 & & & 1 & & -1 \\ & & & & 1 & & 1 & 1 \end{bmatrix},\ b = \begin{bmatrix} 5 \\ 3 \\ 2 \\ -4 \\ -6 \end{bmatrix},\ c = [4,3,2,3,5,2,3,2]^T.$$

Postaviti dualni problem sa dodatnim promenljivama.

$$\xi = -b^T y \to \max$$
$$A^T y + z = c$$
$$z \geq 0$$

Napraviti skicu problema.



Polazeći od pokrivajućeg drveta

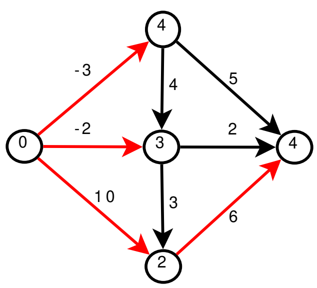
a) $\{ab,ac,ad,de\}$

b) $\{ab,bc,be,cd\}$

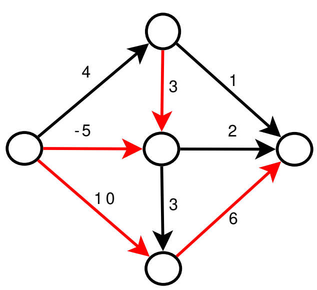
izračunati odgovarajuće protoke, dualne promenljive i dodatne dualne promenljive.

Za oba pokrivajuća drveta: Ako je problem primarno dopustiv - napraviti jednu primarnu pivotizaciju i izračunati cenu transporta u polaznom i dobijenom planu transporta. Ako je dualno dopustiv - napraviti jednu dualnu pivotizaciju.

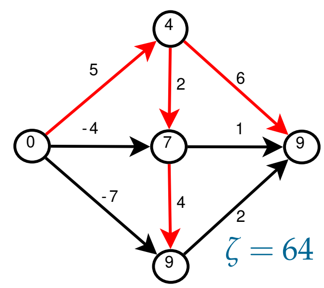
a)



Dualno dopustiv



b)



Primarno dopustiv