Dual od duala

Za problem lineanog programiranja

$$
\begin{array}{lrcrcrcrcl}
 & \zeta & = & 4\,x_1 & + & x_2 & + & 3\,x_3 & \rightarrow & \max \\
 & & & x_1 & + & 4\,x_2 & & & \leq & 1 \\
(9) & & & 3\,x_1 & - & x_2 & + & x_3 & \leq & 3 \\
 & & & \multicolumn{5}{c}{x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0} & &
\end{array}
$$

dual je

$$
\begin{array}{lrcrcrcl}
 & \xi & = & y_1 & + & 3\,y_2 & \rightarrow & \min \\
 & & & y_1 & + & 3\,y_2 & \geq & 4 \\
(10) & & & 4\,y_1 & - & y_2 & \geq & 1 \\
 & & & & & y_2 & \geq & 3 \\
 & & & \multicolumn{5}{c}{y_1 \geq 0, \quad y_2 \geq 0.}
\end{array}
$$

Dual u standardnom obliku je

$$
\begin{array}{lrcrcrcl}
 & -\xi & = & -y_1 & - & 3\,y_2 & \rightarrow & \max \\
 & & & -y_1 & - & 3\,y_2 & \leq & -4 \\
(10a) & & & -4\,y_1 & + & y_2 & \leq & -1 \\
 & & & & - & y_2 & \leq & -3 \\
 & & & \multicolumn{5}{c}{y_1 \geq 0, \quad y_2 \geq 0.}
\end{array}
$$

Prelazak na dual ($x \leftrightarrow y$) daje

$$
\begin{array}{lrcrcrcrcl}
 & -\zeta & = & -4\,x_1 & - & x_2 & - & 3\,x_3 & \rightarrow & \min \\
 & & & -x_1 & - & 4\,x_2 & & & \geq & -1 \\
(9a) & & & -3\,x_1 & + & x_2 & - & x_3 & \geq & -3 \\
 & & & \multicolumn{5}{c}{x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0,} & &
\end{array}
$$

što je ekvivalentno sa (9).

Rečnik u matričnom obliku - izvođenje

(17) $\zeta = c^T \cdot x \to \max$, $A \cdot x = b$, $x \geq 0$ (4)

$A \approx [B \; N]$, B: $m \times m$, N: $m \times n$, $m \times (m+n)$

PERMUT. KOLONA — KOLONE BAZIČNIH PROM. — KOLONE NEBAZ. PROM.

$B^{-1} \cdot B = I_m$

$c = [c_1 \; \dots \; c_n \; 0 \; \dots \; 0]$ PERM. $\to [c_B \; c_N]$, x PERM. $[x_B \; x_N]$

$$(18) \quad \zeta = c^T x = [c_B^T \; c_N^T] \begin{bmatrix} x_B \\ x_N \end{bmatrix} = c_B^T x_B + c_N^T x_N$$

$$(19) \quad b = Ax = [B \; N] \begin{bmatrix} x_B \\ x_N \end{bmatrix} = B x_B + N x_N \quad / \cdot B^{-1}$$

Za matricu B postoji B^{-1}. Rečnik se dobija rešavanjem x_B i ζ preko x_N iz (19) i (18):

$$(19a) \quad B^{-1} b = x_B + B^{-1} N x_N \Rightarrow x_B = B^{-1} b - \left(B^{-1} N\right) x_N \Rightarrow$$

$$(19b) \quad \zeta = c_B^T \left(B^{-1} b - \left(B^{-1} N\right) x_N\right) + c_N^T x_N = c_B^T B^{-1} b - c_B^T \left(B^{-1} N\right) x_N + c_N^T x_N =$$

$$\left(\text{znajući} \quad (A \pm B)^T = A^T \pm B^T \quad \text{i} \quad (AB)^T = B^T A^T \quad \text{i} \quad A^{T^T} = A\right)$$

$$(19c) \quad = c_B^T B^{-1} b - \left(\left(B^{-1} N\right)^T c_B - c_N\right)^T x_N$$

Iz (19c) i (19a) dobijamo

$$(20) \quad \begin{aligned} \zeta &= c_B^T B^{-1} b - \left(\left(B^{-1} N\right)^T c_B - c_N\right)^T x_N \\ x_B &= B^{-1} b - B^{-1} N x_N \,. \end{aligned}$$

Kad uvedemo oznake

$$(21) \quad x_B^* = B^{-1} b, \; \zeta^* = c_B^T B^{-1} b,$$

$$(23) \quad z_N^* = \left(B^{-1} N\right)^T c_B - c_N,$$

dobijamo rečnike primara i duala u matričnom zapisu

NEG. TRANSP.

Primar	Dual
$\zeta = \zeta^* - (z_N^*)^T x_N$	$-\xi = -\zeta^* - (x_B^*)^T z_B$
$x_B = x_B^* - \left(B^{-1} N\right) x_N$	$z_N = z_N^* + \left(B^{-1} N\right)^T z_B$.

(24)

Iz (24) se vidi osobina negativnog transponovanja i dokaz Teoreme jake dualnosti.

$\zeta =$ FUNKCIJA PO x_N

$x_B =$ x_N REČNIK

Permutovanje i transponovanje

REZULTAT MNOŽENJA MATR. OSTAJE ISTI AKO SE KOLONE LEVE MATRICE PERMUTUJU NA ISTI NAČIN KAO VRSTE DESNE MATRICE!

1.

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix} \cdot \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \\ b_{3,1} & b_{3,2} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{1,1}b_{1,1} + a_{1,2}b_{2,1} + a_{1,3}b_{3,1} & a_{1,1}b_{1,2} + a_{1,2}b_{2,2} + a_{1,3}b_{3,2} \\ a_{2,1}b_{1,1} + a_{2,2}b_{2,1} + a_{2,3}b_{3,1} & a_{2,1}b_{1,2} + a_{2,2}b_{2,2} + a_{2,3}b_{3,2} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{1,2}b_{2,1} + a_{1,1}b_{1,1} + a_{1,3}b_{3,1} & a_{1,2}b_{2,2} + a_{1,1}b_{1,2} + a_{1,3}b_{3,2} \\ a_{2,2}b_{2,1} + a_{2,1}b_{1,1} + a_{2,3}b_{3,1} & a_{2,2}b_{2,2} + a_{2,1}b_{1,2} + a_{2,3}b_{3,2} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{1,2} & a_{1,1} & a_{1,3} \\ a_{2,2} & a_{2,1} & a_{2,3} \end{bmatrix} \cdot \begin{bmatrix} b_{2,1} & b_{2,2} \\ b_{1,1} & b_{1,2} \\ b_{3,1} & b_{3,2} \end{bmatrix}$$

2.

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}^{T^T} = \begin{bmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \\ a_{1,3} & a_{2,3} \end{bmatrix}^T = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}$$

$$\left(A^T\right)^T = A$$

3.

$$(A \cdot B)^T = B^T \cdot A^T$$

2×3 3×2 $n\times p$ $p\times m$ $(m\times p \quad p\times m)$ $(m\times n)^T$ $n\times m = n\times m$

$$\left(\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix} \cdot \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \\ b_{3,1} & b_{3,2} \end{bmatrix} \right)^T =$$

$$= \begin{bmatrix} a_{1,1}b_{1,1} + a_{1,2}b_{2,1} + a_{1,3}b_{3,1} & a_{1,1}b_{1,2} + a_{1,2}b_{2,2} + a_{1,3}b_{3,2} \\ a_{2,1}b_{1,1} + a_{2,2}b_{2,1} + a_{2,3}b_{3,1} & a_{2,1}b_{1,2} + a_{2,2}b_{2,2} + a_{2,3}b_{3,2} \end{bmatrix}^T =$$

$$= \begin{bmatrix} a_{1,1}b_{1,1} + a_{1,2}b_{2,1} + a_{1,3}b_{3,1} & a_{2,1}b_{1,1} + a_{2,2}b_{2,1} + a_{2,3}b_{3,1} \\ a_{1,1}b_{1,2} + a_{1,2}b_{2,2} + a_{1,3}b_{3,2} & a_{2,1}b_{1,2} + a_{2,2}b_{2,2} + a_{2,3}b_{3,2} \end{bmatrix} =$$

$$= \begin{bmatrix} b_{1,1}a_{1,1} + b_{2,1}a_{1,2} + b_{3,1}a_{1,3} & b_{1,1}a_{2,1} + b_{2,1}a_{2,2} + b_{3,1}a_{2,3} \\ b_{1,2}a_{1,1} + b_{2,2}a_{1,2} + b_{3,2}a_{1,3} & b_{1,2}a_{2,1} + b_{2,2}a_{2,2} + b_{3,2}a_{2,3} \end{bmatrix} =$$

$$= \begin{bmatrix} b_{1,1} & b_{2,1} & b_{3,1} \\ b_{1,2} & b_{2,2} & b_{3,2} \end{bmatrix} \cdot \begin{bmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \\ a_{1,3} & a_{2,3} \end{bmatrix} =$$

$$= \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \\ b_{3,1} & b_{3,2} \end{bmatrix}^T \cdot \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}^T$$

$$\left(A_{n\times n} B_{n\times n}\right)^{-1} = B^{-1} \cdot A^{-1}$$

Množenje po blokovima

4. Teoreme | ~~Dokaz~~

Identity

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}, C = \begin{bmatrix} i & j \\ k & l \end{bmatrix}, D = \begin{bmatrix} m & n \\ o & p \end{bmatrix}, O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A \cdot C + I \cdot O$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \\ \hline 0 & 0 & e & f \\ 0 & 0 & g & h \end{array}\right] \cdot \left[\begin{array}{cc|cc} i & j & 1 & 0 \\ k & l & 0 & 1 \\ \hline 0 & 0 & m & n \\ 0 & 0 & o & p \end{array}\right] = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \\ \hline 0 & 0 & e & f \\ 0 & 0 & g & h \end{array}\right] \cdot \left[\begin{array}{cc|cc} i & j & 1 & 0 \\ k & l & 0 & 1 \\ \hline 0 & 0 & m & n \\ 0 & 0 & o & p \end{array}\right] = \left[\begin{array}{c|c} A & I \\ \hline O & B \end{array}\right] \cdot \left[\begin{array}{c|c} C & I \\ \hline O & D \end{array}\right] =$$

2x2 2x2 4x4 4x4

$$= \left[\begin{array}{c|c} AC + IO & AI + ID \\ \hline OC + BO & OI + BD \end{array}\right] = \left[\begin{array}{c|c} AC & A + D \\ \hline O & BD \end{array}\right] = \left[\begin{array}{cc|cc} ai + bk & aj + bl & a + m & b + n \\ ci + dk & cj + dl & c + o & d + p \\ \hline 0 & 0 & em + fo & en + fp \\ 0 & 0 & gm + ho & gn + hp \end{array}\right]$$

4x4

Invertovanje matrice

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} 1 & 0 \\ -1/3 & 1/3 \end{bmatrix}}_{A^{-1}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 = A^{-1} \cdot A$$

$(A^{-1})^{-1} = A$

$\boxed{\det A \neq 0}$

$\det(A^{-1}) = (\det A)^{-1}$

$$\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}^{-1} = ?$$

2x2 $(\det A \neq 0)$ A^*

$$\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}^{-1} = (1 \cdot 3 - 1 \cdot 0)^{-1} \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}^T = \frac{1}{3} \begin{bmatrix} 3 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/3 & 1/3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 2 \\ 3 & 2 & 3 \end{bmatrix}^{-1} = ?$$

(ELEMENTARNE TRANSF. SAMO PO VRSTAMA)

$[\underset{n \times n}{A} \mid \underset{n \times n}{I}]$ $n \times (2n)$

$\sim \; \vdots \; \sim$

$[I \mid A^{-1}]$

3x3

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 3 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 2 & -3 & -3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 4 & -2 \\ 0 & 1 & 0 & 0 & -3 & 2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 2 \\ 3 & 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 4 & -2 \\ 0 & -3 & 2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -1/3 & 1/3 \end{array}\right]$$

Data je mreža transporta sa čvorovima $\mathcal{N} = \{a,b,c,d,e\}$ i granama $\mathcal{A} = \{ab,ac,ad,bc,be,cd,ce,de\}$. Zalihe su redom $\{5,3,2,-4,-6\}$, a cene transporta redom $\{4,3,2,3,5,2,3,2\}$.

Problem linearnog programiranja koji odgovara nalaženju najjeftinijeg transporta je $c^T x \to \min$, $Ax = -b$, $x \geq 0$.

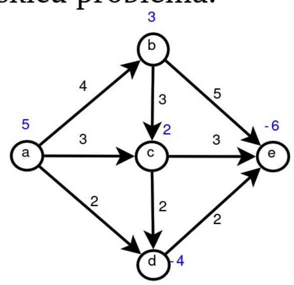
Odrediti matrice A, b i c.

$$A = \begin{bmatrix} -1 & -1 & -1 & & & & & \\ 1 & & & -1 & -1 & & & \\ & 1 & & 1 & & -1 & -1 & \\ & & 1 & & & 1 & & -1 \\ & & & & 1 & & 1 & 1 \end{bmatrix}, \; b = \begin{bmatrix} 5 \\ 3 \\ 2 \\ -4 \\ -6 \end{bmatrix}, \; c = [4,3,2,3,5,2,3,2]^T.$$

Postaviti dualni problem sa dodatnim promenljivama.

$$\xi = -b^T y \to \max$$
$$A^T y + z = c$$
$$z \geq 0$$

Napraviti skicu problema.



Polazeći od pokrivajućeg drveta

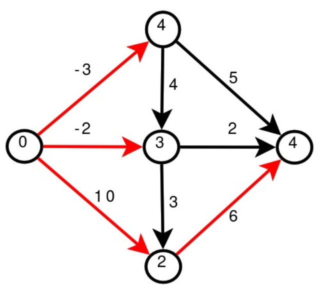
a) $\{ab,ac,ad,de\}$

b) $\{ab,bc,be,cd\}$

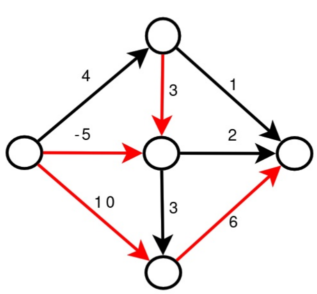
izračunati odgovarajuće protoke, dualne promenljive i dodatne dualne promenljive.

Za oba pokrivajuća drveta: Ako je problem primarno dopustiv - napraviti jednu primarnu pivotizaciju i izračunati cenu transporta u polaznom i dobijenom planu transporta. Ako je dualno dopustiv - napraviti jednu dualnu pivotizaciju.

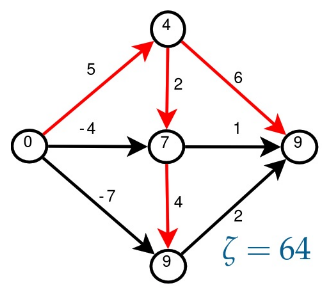
a)



Dualno dopustiv

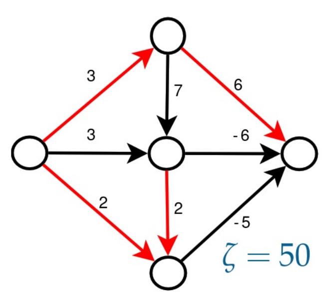


b)



$\zeta = 64$

Primarno dopustiv



$\zeta = 50$