MML Book Exercises rešavanje

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Ovde rešavamo zadatke iz MML Book

MML-book Chapter 2. Linear Algebra, Exercises

Koristićemo R i **matlib** biblioteku.

```
library(matlib)
```

Tekst je u formatu Rmd (R markdown) i može se knit aplikacijom pretvoriti u html (ili pdf) dokument.

2.1

a. Jeste Abelova grupa.

- Zatvorenost: Neka su $x, y \in \mathbb{R}\backslash\{-1\}$. Pretpostavimo da je $a \star b = -1 \Leftrightarrow ab + a + b = -1 \Leftrightarrow a(b+1) = -1(b+1)$.

Poslednja jednakost je moguća samo ako je $a = -1$ ili $b = -1$, dakle, zatvorenost važi.
* Asocijativnost: $(a \star b) \star c = (ab + a + b) \star c = abc + ac + bc + ab + a + b + c$ $a \star (b \star c) = a \star (bc + b + c) = abc + ab + ac + a + bc + b + c$

Vidimo da asocijativnost važi. * Neutralni element je 0: $0 \star b = b$, $a \star 0 = a$. * Inverzni element: $a \star a' = aa' + a + a' = 0 \Leftrightarrow a'(a+1) = -a \Leftrightarrow a' = -a/(a+1)$, što može da se izračuna za sve elemente iz $\mathbb{R}\backslash\{-1\}$. * Komutativnost: očigledna.

b. Rešenja su: $x \in \{-3, 1\}$.

- $3 \star x \star x = 3 \star (x^2 + 2x) = 3x^2 + 6x + 3 + x^2 + 2x = 4x^2 + 8x + 3 = 15 \Leftrightarrow$

$4x^2 + 8x - 12 = 0 \Leftrightarrow x \in \{-3, 1\}$.

2.2

Za $k = 0, 1, \dots, n-1,\ k \in \overline{k}$. Skup $\mathbb{Z}_n = \{\overline{0}, \overline{1}, \dots, \overline{n-1}\}$ predstavlja ostatke pri deljenju sa n. Operacije $\oplus$ i $\otimes$ su sabiranje i množenje po modulu n.

a. Jeste Abelova grupa:

- Zatvorenost: Pošto za svako $k \in \mathbb{Z}$ jedini mogući ostaci pri deljenu sa n su predstavljeni u $\mathbb{Z}_n$, zatvorenost je očigledna.
- Asocijativnost sledi iz asocijativnosti sabiranja: $(\overline{a} \oplus \overline{b}) \oplus \overline{c} = (\overline{a+b}) \oplus \overline{c} = \overline{(a+b)+c} = \overline{a+(b+c)} = \overline{a} \oplus (\overline{b+c}) = \overline{a} \oplus (\overline{b} \oplus \overline{c})$
- Neutralni element je $\overline{0}$: $\overline{0} \oplus \overline{k} = \overline{0+k} = \overline{k} = \overline{k+0} = \overline{k} \oplus \overline{0}$.
- Inverzni element: za $\overline{k}$, inverzni element je $\overline{n-k}$, jer $\overline{k} \oplus \overline{n-k} = \overline{k+n-k} = \overline{n} = \overline{0}$.
- Komutativnost: očigledna.

```r
#########
#  2.2  #
#########

# b
matrix(1:4,ncol=1) %*% matrix(1:4,nrow=1) %% 5
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    2    3    4
## [2,]    2    4    1    3
## [3,]    3    1    4    2
## [4,]    4    3    2    1
```

b. Očigledno je zatvorena operacija, asocijativnost i komutativnost slede, kao i kod $\oplus$ iz asocijativnosti i komutativnosti množenja u skupu $\mathbb{Z}$. Isto, neutralni element za množenje je $\overline{1}$.

Iz tablice se vidi da svaki element ima inverzni: $1^{-1} = 1, 2^{-1} = 3, 3^{-1} = 2, 4^{-1} = 4$.

c. U skupu $\mathbb{Z}_8$ važi $\overline{4} \otimes \overline{2} = \overline{4 \cdot 2} = \overline{8} = \overline{0}$, tako da $(\mathbb{Z}_8 \backslash \{0\}, \otimes)$ nije ni grupoid.

d. $(\mathbb{Z}_n \backslash \{0\}, \otimes)$ je Abelova grupa $\Leftrightarrow$ n je prost broj. Dokaz:

Komutativnost, asocijativnost i neutralnost $\overline{1}$ za $\otimes$ proizilaze iz osobina grupoida $(\mathbb{Z}\backslash\{0\}, \cdot)$.

Ako n nije prost broj, $n = m\ k$, onda $(\mathbb{Z}_n \backslash \{0\}, \otimes)$ nije grupoid jer $\overline{m} \otimes \overline{k} = \overline{n} = \overline{0}$.

Neka je n prost broj i neka $\overline{m} \in \mathbb{Z}_n \backslash \{0\}$ i neka $m \in \{0, 1, \dots, n-1\}$. Brojevi n i m su uzajamno prosti, na osnovu Bezuove teoreme (Bézout's identity) postoje $x, y \in \mathbb{Z}$ tako da $x\ n + y\ m = 1$. Onda je $\overline{y} = \overline{m}^{-1}$, jer

$$\overline{y} \otimes \overline{m} = \overline{y\ m} = \overline{-x\ n + 1} = \overline{1} = \overline{m\ y} = \overline{m} \otimes \overline{y}.$$

2.3

Ako u grupoidu sa neutralnim elementom za podskup važi zatvorenost operacije i nalaženja inverznog elemeta, onda je taj podskup u odnosu na restrikciju operacije grupa. Zatvorenost operacije:

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x_2 & y_2 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x_1 + x_2 & x_1\ z_2 + y_1 + y_2 \\ 0 & 1 & z_1 + z_2 \\ 0 & 0 & 1 \end{bmatrix}$$

Zatvorenost traženja inverznog elementa:

$$\begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -x & x\ z - y \\ 0 & 1 & -z \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -x & x\ z - y \\ 0 & 1 & -z \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$$

Komutativnost ne važi

2.4

Pomnožiti matrice, ako je moguće.

```r
#########
#  2.4  #
#########

# a
# Ne mogu se pomnožiti
```

```r
# b
A = matrix(c(1:9), ncol = 3, byrow = T);
B = matrix(c(1,1,0,0,1,1,1,0,1), ncol = 3, byrow = T); B
```

```
##      [,1] [,2] [,3]
## [1,]    1    1    0
## [2,]    0    1    1
## [3,]    1    0    1
```

```r
A%*%B
```

```
##      [,1] [,2] [,3]
## [1,]    4    3    5
## [2,]   10    9   11
## [3,]   16   15   17
```

```r
# c
B%*%A
```

```
##      [,1] [,2] [,3]
## [1,]    5    7    9
## [2,]   11   13   15
## [3,]    8   10   12
```

```r
# d
A = matrix(c(1,2,1,2,4,1,-1,-4), ncol = 4, byrow = T); A
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    2    1    2
## [2,]    4    1   -1   -4
```

```r
B = matrix(c(0,3,1,-1,2,1,5,2), ncol=2, byrow = T); B
```

```
##      [,1] [,2]
## [1,]    0    3
## [2,]    1   -1
## [3,]    2    1
## [4,]    5    2
```

```r
A%*%B
```

```
##      [,1] [,2]
## [1,]   14    6
## [2,]  -21    2
```

```r
# e
B%*%A
```

```
##      [,1] [,2] [,3] [,4]
## [1,]   12    3   -3  -12
## [2,]   -3    1    2    6
## [3,]    6    5    1    0
## [4,]   13   12    3    2
```

2.5

```r
#########
#  2.5  #
#########
```

```r
# a

A = matrix(c(1,2,2,5,1,5,-1,2,-1,-7,1,-4,-1,-5,3,2),ncol=4); A
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    1   -1   -1
## [2,]    2    5   -7   -5
## [3,]    2   -1    1    3
## [4,]    5    2   -4    2
```

```r
b = c(1,-2,4,6); b
```

```
## [1]  1 -2  4  6
```

```r
gaussianElimination(A,b)
```

```
##      [,1] [,2] [,3]       [,4]        [,5]
## [1,]    1    0    0  0.6666667  1.83333333
## [2,]    0    1    0 -2.6666667 -0.08333333
## [3,]    0    0    1 -1.0000000  0.75000000
## [4,]    0    0    0  0.0000000 -0.50000000
```

```r
# Matrica $A$ i proširena matrica $A|b$ nemaju isti rang, sistem je nemoguć

# b

A = matrix(c(1,1,2,-1,-1,1,-1,2,0,0,0,0,0,-3,1,-2,1,0,-1,-1), ncol = 5); A
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    1   -1    0    0    1
## [2,]    1    1    0   -3    0
## [3,]    2   -1    0    1   -1
## [4,]   -1    2    0   -2   -1
```

```r
b = c(3,6,5,-1); b
```

```
## [1]  3  6  5 -1
```

```r
gaussianElimination(A,b) -> Ab1 ; Ab1
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6]
## [1,]    1    0    0    0   -1    3
## [2,]    0    1    0    0   -2    0
## [3,]    0    0    0    1   -1   -1
## [4,]    0    0    0    0    0    0
```

```r
Ab2=rbind(Ab1[1:2,],-(1:6==3),Ab1[3,],-(1:6==5)); Ab2
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6]
## [1,]    1    0    0    0   -1    3
## [2,]    0    1    0    0   -2    0
## [3,]    0    0   -1    0    0    0
## [4,]    0    0    0    1   -1   -1
## [5,]    0    0    0    0   -1    0
```

Rešenje sistema su svi vektori $\boldsymbol{x} = [3,0,0,-1,0]^T + x_3[0,0,1,0,0]^T + x_5[1,2,0,1,1]^T$, $x_3, x_5 \in \mathbb{R}$.

2.6

```
#########
#  2.6  #
#########

A=matrix(c(0,1,0,0,1,0,0,0,0,1,1,0,0,1,0,0,0,1), ncol = 6, byrow = TRUE); A
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6]
## [1,]    0    1    0    0    1    0
## [2,]    0    0    0    1    1    0
## [3,]    0    1    0    0    0    1
```

```
b=c(2,-1,2); b
```

```
## [1]  2 -1  2
```

```
gaussianElimination(A,b) -> Ab1 ; Ab1
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6] [,7]
## [1,]    0    1    0    0    0    1    2
## [2,]    0    0    0    1    0    1   -1
## [3,]    0    0    0    0    1   -1    0
```

```
Ab2=rbind(-(1:7==1),Ab1[1,],-(1:7==3),Ab1[2:3,],-(1:7==6)); Ab2
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6] [,7]
## [1,]   -1    0    0    0    0    0    0
## [2,]    0    1    0    0    0    1    2
## [3,]    0    0   -1    0    0    0    0
## [4,]    0    0    0    1    0    1   -1
## [5,]    0    0    0    0    1   -1    0
## [6,]    0    0    0    0    0   -1    0
```

Rešenje sistema su svi vektori $\boldsymbol{x} = [0,2,0,-1,0,0]^T + x_1[1,0,0,0,0,0]^T + x_3[0,0,1,0,0,0]^T + x_6[0,1,0,1,-1,-1]^T$, $x_1, x_3, x_6 \in \mathbb{R}$.

2.7

```
#########
#  2.7  #
#########

A = matrix(c(6,4,3,6,0,9,0,8,0), ncol = 3, byrow = TRUE); A
```

```
##      [,1] [,2] [,3]
## [1,]    6    4    3
## [2,]    6    0    9
## [3,]    0    8    0
```

```
A = A - 12 * diag(3); A
```

```
##      [,1] [,2] [,3]
## [1,]   -6    4    3
## [2,]    6  -12    9
## [3,]    0    8  -12
```

```r
A = rbind(A,c(1,1,1)); A
```

```
##      [,1] [,2] [,3]
## [1,]   -6    4    3
## [2,]    6  -12    9
## [3,]    0    8  -12
## [4,]    1    1    1
```

```r
b = c(0,0,0,1); b
```

```
## [1] 0 0 0 1
```

```r
gaussianElimination(A,b) -> Ab1 ; Ab1
```

```
##      [,1] [,2] [,3]  [,4]
## [1,]    1    0    0 0.375
## [2,]    0    1    0 0.375
## [3,]    0    0    1 0.250
## [4,]    0    0    0 0.000
```

Rešenje: $\boldsymbol{x} = [0.375, 0.375, 0.250]^T = [3/8, 3/8, 2/8]^T$.

2.8

```r
#########
#  2.8  #
#########

# a
A = matrix(c(2,3,4,3,4,5,4,5,6), ncol = 3, byrow = T); A
```

```
##      [,1] [,2] [,3]
## [1,]    2    3    4
## [2,]    3    4    5
## [3,]    4    5    6
```

```r
A = cbind(A,diag(3)); A
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6]
## [1,]    2    3    4    1    0    0
## [2,]    3    4    5    0    1    0
## [3,]    4    5    6    0    0    1
```

```r
gaussianElimination(A) -> A1; A1
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6]
## [1,]    1    0   -1    0   -5    4
## [2,]    0    1    2    0    4   -3
## [3,]    0    0    0    1   -2    1
```

```r
# Nema inverznu, rang matrice je 2

# b
A = matrix(c(1,0,1,0,0,1,1,0,1,1,0,1,1,1,1,0), ncol = 4, byrow = T); A
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    0    1    0
## [2,]    0    1    1    0
```

```
## [3,]    1    1    0    1
## [4,]    1    1    1    0
```

```r
A0 = cbind(A,diag(4)); A0
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8]
## [1,]    1    0    1    0    1    0    0    0
## [2,]    0    1    1    0    0    1    0    0
## [3,]    1    1    0    1    0    0    1    0
## [4,]    1    1    1    0    0    0    0    1
```

```r
gaussianElimination(A0) -> A1; A1
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8]
## [1,]    1    0    0    0    0   -1    0    1
## [2,]    0    1    0    0   -1    0    0    1
## [3,]    0    0    1    0    1    1    0   -1
## [4,]    0    0    0    1    1    1    1   -2
```

```r
A1 = A1[,5:8]; A1
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    0   -1    0    1
## [2,]   -1    0    0    1
## [3,]    1    1    0   -1
## [4,]    1    1    1   -2
```

```r
A %*% A1
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    0    0    0
## [2,]    0    1    0    0
## [3,]    0    0    1    0
## [4,]    0    0    0    1
```

```r
library(xtable)
print(xtable(A1, digits = 0), floating=FALSE, tabular.environment="bmatrix", hline.after=NULL, include.
```

```
## % latex table generated in R 4.5.0 by xtable 1.8-4 package
## % Thu Dec  4 21:27:17 2025
## \begin{bmatrix}{rrrr}
##   0 & -1 & 0 & 1 \\
##   -1 & 0 & 0 & 1 \\
##   1 & 1 & 0 & -1 \\
##   1 & 1 & 1 & -2 \\
##   \end{bmatrix}
```

Rešenje: $A^{-1} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & -2 \end{bmatrix}$.

2.9

a. Da. span $([1,1,1]^T, [0,1,-1]^T) = \lambda[1,1,1]^T + \mu^3[0,1,-1]^T, \lambda, \mu \in \mathbb{R}$
b. Ne. Nije grupa.
c. Da ako i samo ako $\gamma = 0$.
d. Ne. (Homogenost!)

2.10

```r
#########
#  2.10 #
#########

# a
A = matrix(c(2,-1,3,1,1,-2,3,-3,8), ncol = 3); A
```

```
##      [,1] [,2] [,3]
## [1,]    2    1    3
## [2,]   -1    1   -3
## [3,]    3   -2    8
```

```r
gaussianElimination(A)
```

```
##      [,1] [,2] [,3]
## [1,]    1    0    2
## [2,]    0    1   -1
## [3,]    0    0    0
```

```r
# b
A = matrix(c(1,2,1,0,0,1,1,0,1,1,1,0,0,1,1), ncol = 3); A
```

```
##      [,1] [,2] [,3]
## [1,]    1    1    1
## [2,]    2    1    0
## [3,]    1    0    0
## [4,]    0    1    1
## [5,]    0    1    1
```

```r
gaussianElimination(A)
```

```
##      [,1] [,2] [,3]
## [1,]    1    0    0
## [2,]    0    1    0
## [3,]    0    0    1
## [4,]    0    0    0
## [5,]    0    0    0
```

a. Linearno zavisni: $\boldsymbol{x}_3 = 2\boldsymbol{x}_1 - \boldsymbol{x}_2$
b. Linearno nezavisni, rang $([\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3]) = 3$.

2.11

```r
#########
#  2.11 #
#########

A = matrix(c(1,1,1,1,2,3,2,-1,1), ncol = 3); A
```

```
##      [,1] [,2] [,3]
## [1,]    1    1    2
## [2,]    1    2   -1
## [3,]    1    3    1
```

```r
y = c(1,-2,5)
gaussianElimination(A,y)->A1; A1
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    0    0   -6
## [2,]    0    1    0    3
## [3,]    0    0    1    2
```

$\boldsymbol{y} = -6\boldsymbol{x}_1 + 3\boldsymbol{x}_2 + 2\boldsymbol{x}_3$.

2.12

```r
#########
#  2.12 #
#########

A = matrix(c(1,1,-3,1,2,-1,0,-1,-1,1,-1,1), ncol = 3); A
```

```
##      [,1] [,2] [,3]
## [1,]    1    2   -1
## [2,]    1   -1    1
## [3,]   -3    0   -1
## [4,]    1   -1    1
```

```r
gaussianElimination(A)
```

```
##      [,1] [,2]       [,3]
## [1,]    1    0  0.3333333
## [2,]    0    1 -0.6666667
## [3,]    0    0  0.0000000
## [4,]    0    0  0.0000000
```

```r
B = matrix(c(-1,-2,2,1,2,-2,0,0,-3,6,-2,-1), ncol = 3); B
```

```
##      [,1] [,2] [,3]
## [1,]   -1    2   -3
## [2,]   -2   -2    6
## [3,]    2    0   -2
## [4,]    1    0   -1
```

```r
gaussianElimination(B)
```

```
##      [,1] [,2] [,3]
## [1,]    1    0   -1
## [2,]    0    1   -2
## [3,]    0    0    0
## [4,]    0    0    0
```

```r
C = cbind( A[,1:2], -B[,1:2]); C
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    2    1   -2
## [2,]    1   -1    2    2
## [3,]   -3    0   -2    0
## [4,]    1   -1   -1    0
```

```r
gaussianElimination(C) -> C1; C1
```

```
##      [,1] [,2] [,3]       [,4]
## [1,]    1    0    0 -0.4444444
## [2,]    0    1    0 -1.1111111
```

```
## [3,]    0    0    1  0.6666667
## [4,]    0    0    0  0.0000000
```

```r
C2 = rbind(C1[1:3,],-(1:4==4))

D=matrix(round(-9*C2[,4]),ncol = 1); D
```

```
##      [,1]
## [1,]    4
## [2,]   10
## [3,]   -6
## [4,]    9
```

```r
C %*% D
```

```
##      [,1]
## [1,]    0
## [2,]    0
## [3,]    0
## [4,]    0
```

```r
(C[,1:2] %*% D[1:2,])
```

```
##      [,1]
## [1,]   24
## [2,]   -6
## [3,]  -12
## [4,]   -6
```

```r
(C[,1:2] %*% D[1:2,])/6
```

```
##      [,1]
## [1,]    4
## [2,]   -1
## [3,]   -2
## [4,]   -1
```

```r
(C[,3:4] %*% -D[3:4,])
```

```
##      [,1]
## [1,]   24
## [2,]   -6
## [3,]  -12
## [4,]   -6
```

```r
(C[,3:4] %*% -D[3:4,])/6
```

```
##      [,1]
## [1,]    4
## [2,]   -1
## [3,]   -2
## [4,]   -1
```

$U_1 \cap U_2 = \text{span}([4, -1, -2, -1]^T)$

2.13

```r
#########
#  2.13 #
```

```r
#########

A = matrix(c(1,1,2,1,0,-2,1,0,1,-1,3,1),ncol=3); A
```

```
##      [,1] [,2] [,3]
## [1,]    1    0    1
## [2,]    1   -2   -1
## [3,]    2    1    3
## [4,]    1    0    1
```

```r
gaussianElimination(A)
```

```
##      [,1] [,2] [,3]
## [1,]    1    0    1
## [2,]    0    1    1
## [3,]    0    0    0
## [4,]    0    0    0
```

```r
A=matrix(c(3,1,7,3,-3,2,-5,-1,0,3,2,2),ncol=3); A
```

```
##      [,1] [,2] [,3]
## [1,]    3   -3    0
## [2,]    1    2    3
## [3,]    7   -5    2
## [4,]    3   -1    2
```

```r
gaussianElimination(A)
```

```
##      [,1] [,2] [,3]
## [1,]    1    0    1
## [2,]    0    1    1
## [3,]    0    0    0
## [4,]    0    0    0
```

$U_1 \cap U_2 = \text{span}\,([1, 1, -1]^T)$.

2.14

```r
#########
#  2.14 #
#########

A = matrix(c(1,1,2,1,0,-2,1,0,1,-1,3,1), ncol = 3); A
```

```
##      [,1] [,2] [,3]
## [1,]    1    0    1
## [2,]    1   -2   -1
## [3,]    2    1    3
## [4,]    1    0    1
```

```r
gaussianElimination(A)
```

```
##      [,1] [,2] [,3]
## [1,]    1    0    1
## [2,]    0    1    1
## [3,]    0    0    0
## [4,]    0    0    0
```

```r
B = matrix(c(3,1,7,3,-3,2,-5,-1,0,3,2,2),ncol=3,byrow = F); B
```

```
##      [,1] [,2] [,3]
## [1,]    3   -3    0
## [2,]    1    2    3
## [3,]    7   -5    2
## [4,]    3   -1    2
```

```r
gaussianElimination(B)
```

```
##      [,1] [,2] [,3]
## [1,]    1    0    1
## [2,]    0    1    1
## [3,]    0    0    0
## [4,]    0    0    0
```

```r
C = cbind( A[,1:2],-B[,1:2]); C
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    0   -3    3
## [2,]    1   -2   -1   -2
## [3,]    2    1   -7    5
## [4,]    1    0   -3    1
```

```r
gaussianElimination(C)
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    0   -3    0
## [2,]    0    1   -1    0
## [3,]    0    0    0    1
## [4,]    0    0    0    0
```

```r
D=matrix(c(3,1,1,0), ncol = 1); D
```

```
##      [,1]
## [1,]    3
## [2,]    1
## [3,]    1
## [4,]    0
```

```r
C %*% D
```

```
##      [,1]
## [1,]    0
## [2,]    0
## [3,]    0
## [4,]    0
```

```r
C[,1:2] %*% D[1:2,]
```

```
##      [,1]
## [1,]    3
## [2,]    1
## [3,]    7
## [4,]    3
```

$U_1 \cap U_2 = \mathrm{span}\left([3,1,7,3]^T\right)$.

2.15

a. F je podprostor jer jer skup rešenja homogenog sistema. G je podprostor jer je $G = \text{span}\left([1,1,1]^T, [-1,1,-3]^T\right)$.
b. Neka $[x,y,z]^T \in F \cap G$. Onda $x = a-b, y = a+b, z = a-3b$.

$x+y-z=0 \iff a-b+a+b-(a-3b)=0 \iff a=-3b \iff x=-4b, y=-2b, z=-6b$ Dakle, $F \cap G = \text{span}([2,1,3]^T)$ c. Nađimo prvo bazu F koristeći -1-trik:

```
#########
#  2.15 #
#########

# c.
A=rbind(c(1,1,-1),-(1:3==2),-(1:3==3)); A
```

```
##      [,1] [,2] [,3]
## [1,]    1    1   -1
## [2,]    0   -1    0
## [3,]    0    0   -1
```

```
B=cbind(c(1,1,1),c(-1,1,-3)); B
```

```
##      [,1] [,2]
## [1,]    1   -1
## [2,]    1    1
## [3,]    1   -3
```

```
C=cbind(A[,2:3],-B); C
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1   -1   -1    1
## [2,]   -1    0   -1   -1
## [3,]    0   -1   -1    3
```

```
gaussianElimination(C)->C1; C1
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    0    0   -2
## [2,]    0    1    0   -6
## [3,]    0    0    1    3
```

```
C2 = rbind(C1[1:3,],-(1:4==4))
D = C2[,4]
C %*% D
```

```
##      [,1]
## [1,]    0
## [2,]    0
## [3,]    0
```

```
A[,2:3] %*% D[1:2]
```

```
##      [,1]
## [1,]    4
## [2,]    2
## [3,]    6
```

```
A[,2:3] %*% D[1:2]/2
```

```
##      [,1]
## [1,]    2
## [2,]    1
## [3,]    3
```

$F \cap G = \text{span}\left([2,1,3]^T\right)$.

2.16

a. Da. $\forall f,g \in L^1([a,b]),\ \forall\lambda,\psi \in \mathbb{R}\int_a^b(\lambda f(x)+\psi g(x))dx = \lambda\ \int_a^b f(x)dx + \psi\int_a^b g(x)dx$. Vektorski prostor $L^1([a,b])$ je beskonačno-dimenzionalni vektorski prostor nad $\mathbb{R}$.

b. Da. $\forall f,g \in C^1, \forall\lambda,\psi \in \mathbb{R}, \forall x \in \mathbb{R}\ (\lambda f(x)+\psi g(x))' = \lambda f'(x) + g'(x)$. C^1 i C^0 su beskonačno-dimenzionalni vektorski prostori nad $\mathbb{R}$.

c. Ne. $\Phi(0) = 1$.

d. Da. Na osnovu distributivnosti množenja matrica i osobina množenja matrice skalarom.

e. Da. Na osnovu distributivnosti množenja matrica i osobina množenja matrice skalarom. Ova transformacija predstavlja rotaciju za ugao θ u $\mathbb{R}^2$.

2.17

```
#########
#  2.17 #
#########

A=matrix(c(3,2,1,1,1,1,1,-3,0,2,3,1), ncol = 3, byrow = T); A
```

```
##      [,1] [,2] [,3]
## [1,]    3    2    1
## [2,]    1    1    1
## [3,]    1   -3    0
## [4,]    2    3    1
```

```
gaussianElimination(A)->A1; A1
```

```
##      [,1] [,2] [,3]
## [1,]    1    0    0
## [2,]    0    1    0
## [3,]    0    0    1
## [4,]    0    0    0
```

- $A_\Phi = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix}$
- rk $(A_\Phi) = 3$
- $\ker(\Phi) = \{0\}, \text{Im}(\Phi) = \text{span}([1,0,0,0]^T, [0,1,0,0]^T, [0,0,1,0]^T)$, $\dim(\ker(\Phi)) = 0, \dim(\text{Im}(\Phi)) = 3$.

2.19

```
#########
#  2.19 #
```

```r
#########

A=matrix(c(1,1,1,1,-1,1,0,0,1), ncol = 3); A
```

```
##      [,1] [,2] [,3]
## [1,]    1    1    0
## [2,]    1   -1    0
## [3,]    1    1    1
```

```r
B=matrix(c(1,1,1,1,2,1,1,0,0), ncol = 3); B
```

```
##      [,1] [,2] [,3]
## [1,]    1    1    1
## [2,]    1    2    0
## [3,]    1    1    0
```

```r
inv(B) %*% A %*% B
```

```
##      [,1] [,2] [,3]
## [1,]    6    9    1
## [2,]   -3   -5    0
## [3,]   -1   -1    0
```

$\tilde{A}_\Phi = \begin{bmatrix} 6 & 9 & 1 \\ -3 & -5 & 0 \\ -1 & -1 & 0 \end{bmatrix}$

2.20

Neka su V i W konačnodimenzionalni vektorski prostori. Neka su $B = (b_1, \ldots, b_n)$ uređena baza od V i $C = (c_1, \ldots, c_m)$ uređena baza od W. Neka je A_Φ matrica linearne transformacije $\Phi : V \to W$ u odnosu na baze B i C.

Neka su $\tilde{B} = (\tilde{b}_1, \ldots, \tilde{b}_n)$ uređena baza od V i $\tilde{C} = (\tilde{c}_1, \ldots, \tilde{c}_m)$ uređena baza od W. Neka su matrice $S = [\tilde{b}_1, \ldots, \tilde{b}_n]$ i $T = [\tilde{c}_1, \ldots, \tilde{c}_m]$.

Onda se matrica linearne transformacije Φ u odnosu na baze $\tilde{B}$ i $\tilde{C}$ dobija formulom $\tilde{A}_\Phi = T^{-1} A_\Phi S$.

```r
#########
#  2.20 #
#########

# a
B1 = matrix(c(2,1,-1,-1), ncol = 2); B1
```

```
##      [,1] [,2]
## [1,]    2   -1
## [2,]    1   -1
```

```r
gaussianElimination(B1)
```

```
##      [,1] [,2]
## [1,]    1    0
## [2,]    0    1
```

```r
B2 = matrix(c(2,-2,1,1), ncol = 2); B2
```

```
##      [,1] [,2]
## [1,]    2    1
```

```
## [2,]   -2    1
```

```r
gaussianElimination(B2)
```

```
##      [,1] [,2]
## [1,]    1    0
## [2,]    0    1
```

Vidimo da je $\mathrm{rk}([\boldsymbol{b}_1, \boldsymbol{b}_2]) = \mathrm{rk}([\boldsymbol{b}_1', \boldsymbol{b}_2']) = 2$, znači da su i B i B' baze od $\mathbb{R}^2$.

Neka je $\mathrm{id}_{\mathbb{R}^2}$ identična linearna transformacija. $\mathrm{id}_{\mathbb{R}^2} : \boldsymbol{x} \mapsto \boldsymbol{x}$.

Onda su matrice $[\boldsymbol{b}_1, \boldsymbol{b}_2]$ i $[\boldsymbol{b}_1', \boldsymbol{b}_2']$ matrice transformacije $\mathrm{id}_{\mathbb{R}^2}$ u odnosu na standardnu bazu i redom bazu B i B'.

Vektor $\boldsymbol{x}$ izražen u bazi B kao, recimo, $\boldsymbol{x} = 2\boldsymbol{b}_1 + 3\boldsymbol{b}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}_B$ se u standardnoj bazi dobija kao $[\boldsymbol{b}_1, \boldsymbol{b}_2] \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Obrnuto, ako je $\boldsymbol{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ vektor u standardnoj bazi i želimo da ga prikažemo u bazi B, onda dobijamo $\boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}_B$. Račun: $\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

```r
#########
#  2.20 #
#########

# b
P1=inv(B1) %*% B2; P1
```

```
##      [,1] [,2]
## [1,]    4    0
## [2,]    6   -1
```

Kad želimo da nađemo matricu promene iz baze B u bazu B', onda tražimo prikazivanje identične transformacije u odnosu na baze B i B': $P_1 = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix}$

```r
#########
#  2.20 #
#########

# c
P2 = matrix(c(1,2,-1,0,-1,2,1,0,-1), ncol = 3); P2
```

```
##      [,1] [,2] [,3]
## [1,]    1    0    1
## [2,]    2   -1    0
## [3,]   -1    2   -1
```

```r
det(P2)
```

```
## [1] 4
```

Neka je matrica $P_2 = [\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3]$.

Pošto je $\det P_2 \neq 0$, C jeste baza, jer je punog ranga. Upravo matrica P_2 je matrica promene iz baze C u standardnu bazu C'.

```r
#########
#  2.20 #
#########

# d
AB = matrix(c(1,1,1,-1,0,2,1,-1,1,3), ncol = 5); AB
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    1    1    0    1    1
## [2,]    1   -1    2   -1    3
```

```r
ABG = gaussianElimination(AB); ABG
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    1    0    1    0    2
## [2,]    0    1   -1    1   -1
```

```r
A = t(ABG[,3:5]); A
```

```
##      [,1] [,2]
## [1,]    1   -1
## [2,]    0    1
## [3,]    2   -1
```

$\Phi(\boldsymbol{b}_1 + \boldsymbol{b}_2) = \Phi(\boldsymbol{b}_1) + \Phi(\boldsymbol{b}_1) = \qquad c_2 + c_3$

$\Phi(\boldsymbol{b}_1 - \boldsymbol{b}_2) = \Phi(\boldsymbol{b}_1) - \Phi(\boldsymbol{b}_1) = 2c_1 - c_2 + 3c_3$

Primenom Gausovih transformacija:

$\Phi(\boldsymbol{b}_1) = \quad c_1 \qquad + 2c_3$

$\Phi(\boldsymbol{b}_2) = -c_1 + c_2 - \quad c_3$

Dobijamo da je $A_\Phi = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}$ matrica transformacija ϕ u odnosu na uređene baze $B = (\boldsymbol{b}_1, \boldsymbol{b}_2)$ i $C = (\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3)$.

```r
#########
#  2.20 #
#########

# e
A1 = P2 %*% A%*% P1; A1
```

```
##      [,1] [,2]
## [1,]    0    2
## [2,]  -10    3
## [3,]   12   -4
```

Koristimo formulu $\tilde{A}_\Phi = T^{-1} A_\Phi S$, gde je $T^{-1} = P_2$ (dobijeno pod c) i $S = P_1$ (dobijeno pod b).

```r
#########
#  2.20 #
#########

# f
x = matrix(c(2,3), ncol=1); x
```

```
##      [,1]
```

```
## [1,]    2
## [2,]    3
```

```r
# (i)
P1 %*% x
```

```
##      [,1]
## [1,]    8
## [2,]    9
```

```r
# (ii)
A %*% P1 %*% x
```

```
##      [,1]
## [1,]   -1
## [2,]    9
## [3,]    7
```

```r
# (iii)
P2 %*% A %*% P1 %*% x
```

```
##      [,1]
## [1,]    6
## [2,]  -11
## [3,]   12
```

```r
# (iv)
A1 %*% x
```

```
##      [,1]
## [1,]    6
## [2,]  -11
## [3,]   12
```